1. Suppose that V, W are (possibly infinite-dimensional) vector spaces with norms $\|*\|_V, \|*\|_W$, giving a topology defined by the metric $d_V(x, y) = \|x - y\|_V$(similarly for W). Suppose further that W is complete.
Suppose that $H \subseteq V$ is a vector subspace, and is dense in the metric topology. Show that if $L : H \to W$ is linear, with $\|Lx\|_W \leq C \|x\|_V$ for some constant $C > 0$ independent of $x \in H$, then there exists a unique extension to V of L, with the same bound.

2. We define a distribution as follows:
Suppose that f_n, f are smooth functions with compact support(i.e. $\in C_c^\infty(U)$) in an open set $U \subseteq \mathbb{R}^n$, and that all have support contained in the same compact set $K \subseteq U$.
Suppose further that all derivatives of all orders of f_n converge to those of f(uniformly). We then say that $f_n \to f$ in $C_c^\infty(U)$. A distribution is a linear map $\phi : C_c^\infty(U) \to \mathbb{R}$ such that $f_n \to f$ in $C_c^\infty(U)$ implies $\phi(f_n) \to \phi(f)$.

   (a) Show that given $g \in C(U)$, it holds that $\phi_g(f) = \int_U fg dx$ is a distribution. Similarly, $D^\alpha \phi_g(f) = (-1)^{|\alpha|} \int_U (D^\alpha f) g dx$ are distributions.

   (b) Use integration-by-parts(or the divergence theorem) to show that if $f, g \in C_c^\infty(U)$ then
   $$D^\alpha \phi_g(f) = \phi_{D^\alpha g}(f)$$

   (c) Use Hoelder's inequality(Theorem 6.2 in Folland) to show that given $g \in L^2(U)$, it holds that $\phi_g(f) = \int_U fg dx$ is a distribution

   (d) Use Problem 1, and Folland theorem 6.15 to show that if $g \in L^2(U)$ and if
   $$\|D^\alpha \phi_g(f)\|_{L^2(U)} \leq C \|f\|_{L^2(U)}$$
   for some $C > 0$ then
   $$D^\alpha \phi_g(f) = \int_U fh dx$$
   for some $h \in L^2(U)$. You may assume that $C_c^\infty(U)$ is dense in $L^2(U)$.

   (e) (Harder) Suppose that $g, h \in C(U)$, and suppose that
   $$\phi_g(f) = \phi_h(f), \forall f \in C_c^\infty(U)$$
   then $h = g$. (Hint: Suppose that $h(x) - g(x) > 0$, then $\{y : h(y) - g(y) > 0\}$ is open and non-empty, and recall the construction of a partition of unity)